

Errata and Additional Theory to Clarify Some Statements

WYLE LABORATORIES-RESEARCH STAFF

REPORT WR 66-36

DYNAMIC RESPONSE OF BUCKLED STIFFENED
PANELS TYPICAL OF S-II STAGE FORWARD
SKIRT UNDER ACOUSTIC LOADS

by

C.L. Amba-Rao

Final Report Submitted under
Brown Engineering Co. Work Order No. 933-50-19-9150
Technical Directive S-5 (Phase 1)
Sub Contract 20073-01

Page

- iv Line 3 from bottom, replace (39) by (42).
Line 2 from bottom, replace (43) by (46).
- vii Line 13 from top, add "(tension force is positive, and compressive force force is negative)".
- 11 Line 1 from top, replace "unloaded," by "freely".
Add after equation (28), "The physical interpretation of the boundary conditions is as follows:
The boundary conditions represent moment free support in transverse bending with membrane action being resisted by stiffeners (at the edges) rigid along their own axis (and thus capable of taking membrane shear loads
$$\frac{\partial^2 F}{\partial x \partial y})$$
, but non-resistant towards loads perpendicular to their axes and
in the x, y - plane".
(Reissner, E. "Non-linear Effects in Vibrations of Cylindrical Shells", The Ramo-Wooldridge Corporation Rep. No. AM 5-6, 1955, p. 15).
Line 5 from top, add "with respect to x " after sine transforms .

Page

11 Line 1 from bottom, replace " \bar{F}_{22} and $\bar{F}_{22} \dots$ " by " \bar{F}_{21} and $\bar{F}_{22} \dots$ ".

14 Lines 2 to 6 from the top should read as

$$\begin{aligned} 5 E_1 &= - 5.5458 \dots \times 10^8 A_{41}^2 \\ 5 E_2 &= 5.9363 \dots \times 10^9 A_{41}^2 \\ 5 E_3 &= 7.2174 \dots \times 10^{10} A_{41}^2 \\ 5 E_4 &= - 1.7082 \dots \times 10^7 A_{41}^2 \\ 5 E_5 &= 7.4502 \dots \times 10^{10} A_{41}^2 \end{aligned}$$

Line 7 from top should read as

$$5 E_6 = 1.763377574272 \times 10^7 A_{41}^2$$

Line 2 from bottom, add ")" at end.

15 Equation (47) now should read as

$$A_{41} \left[(303.05 - 314.42 k) + \frac{1}{5} (-107 + 437 - 5.44) 10^6 A_{41}^2 \right] = 0.$$

Line 2 from bottom, replace 0.000175 by $0.000175\sqrt{5}$.

Line 1 from bottom, replace 0.0044 by $0.0044\sqrt{5}$.

16 Equation (49) now should read as

$$A_{41} \left[(155.16 - 160.98 k) + \frac{1}{5} (-85.7 + 349.7 - 4.35) 10^6 A_{41}^2 \right] = 0.$$

Line 8 from bottom, replace $\frac{w_{\max}}{h}$ by $\frac{w_{\max}}{\sqrt{5} h}$.

In equation (50) replace N_x by N_{xi} .

Page

17 Line 26 from top, replace "compression force N_x " by "force N_{xi} ".

Add the following sentences at the beginning of Section 3.2, for clarification.

"In Sections 3.2 to 3.4, both inclusive, the analysis presented is for N_{xi} , assuming the edgewise force is tension. If N_{xi} is compressive, as it is for the problem in hand, replace N_{xi} by $-N_{xi}$ ".

23 Add at the bottom of the page.

"If m (number of half waves in x direction) = 4 and n (number of half waves in y direction) = 1, and sections of the buckled plate on $y = 0$

and $x = \frac{a}{4} m$ (m , an integer < 4), then $D_x = D_y$ (under the assumption that the boundary conditions at the four edges of the square panel of side b are alike) ($M_{xy} = M_{yx}$), and $D_{xy} = D_{yx}$. It follows automatically that the neutral surface in the buckled position coincides with the neutral surface of the undeflected plate".

24 Line 3 from top, replace "simply" by "freely".

38 Line 2 from top, replace (39) by (42).

Line 4 from top, read $E = 10 \times 10^6$.

40 Line 2 from top, replace (43) by (46).

41 Add the following after equation (G5).

"Note

$$w_{xx} = - \left(\frac{4\pi}{a} \right)^2 \sin \frac{4\pi x}{a} \cos \frac{\pi y}{b}$$

and

$$N_{xi} = - k \frac{3.84 E h^3}{b^2} \text{ as } N_{xi} \text{ is a compressive force. If the}$$

above values are substituted in the term $-N_{xi} w_{xx}$ of (G4), one gets

$$- A_{41} \left(\frac{4\pi}{a} \right)^2 \sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} \left(\frac{k 3.84 E h^3}{b^2} \right) \text{ of (G5) .}$$

Page

44 Line 11 from top, replace -107.1129 by $-\frac{107.1129}{5}$ and
 -856.9028 by $-\frac{856.9028}{5}$.

 Line 12 from top, replace 437.1314 by $\frac{437.1314}{5}$ and
 349.7051 by $\frac{349.7051}{5}$.

 Line 13 from top, replace -544.2502 by $-\frac{544.2502}{5}$ and
 -435.4001 by $-\frac{435.4001}{5}$.

45 Line 5 from top, replace $\frac{w}{h}$ by $\frac{w}{\sqrt{5}h}$.

49 Line 3 from bottom, replace $\frac{w}{h}$ by $\frac{w}{\sqrt{5}h}$.

WYLE LABORATORIES - RESEARCH STAFF

REPORT WR 66-36

DYNAMIC RESPONSE OF BUCKLED STIFFENED
PANELS TYPICAL OF S-II STAGE FORWARD
SKIRT UNDER ACOUSTIC LOADS

Final Report Submitted under
Brown Engineering Co. Work Order No. 933-50-19-9150
Technical Directive S-5 (Phase 1)
Sub Contract 20073-01

Prepared by

C. L. Amba-Rao
C.L. Amba-Rao

Approved by

B. J. Spice
B. J. Spice

Approved by

Kenneth McK. Eldred
Kenneth McK. Eldred
Director of Research

Date June, 1966

COPY NO. _____

SUMMARY

11692

A convenient method is presented to analyze the problem of a buckled panel, typical of the S - II stage forward skirt, under acoustic loads. It is proved that the total response (deflection) w consists of two components, the static and the dynamic, and superposition in a modified form is valid. The static component of the displacement of the deflection surface in the post buckled state is calculated, using von Kármán equations, and Galerkin method. The dynamic component of response is calculated formally, utilizing the concept of equivalent fictitious uniformly thick anisotropic plates for small vibrations due to acoustic loadings. Having obtained w , the method of analysis to obtain the principal strains, is formulated.

AUTHOR

TABLE OF CONTENTS

	Page
SUMMARY	ii
TABLE OF CONTENTS	iii
LIST OF FIGURES	v
LIST OF TABLES	vi
NOTATION	vii
 1.0 INTRODUCTION	 1
1.1 Literature Survey	1
1.2 Method of Approach	1
1.2.1 Superposition in a Modified Form of Beam-column Loadings and Responses	2
 2.0 MATHEMATICAL FORMULATION OF THE PROBLEM- STATIC DEFLECTION OF THE BUCKLED PLATE	 5
2.1 Method of Solution of Buckled Plate in Postbuckled State	6
2.2 Calculation of Stress Function	6
2.2.1 First Approximation	6
2.2.2 Fourier Finite Sine Transform of the Nonhomogeneous Partial Differential Equation	8
2.2.3 Complete Formal Solution of Stress Function	9
2.2.4 Application to a Specific Problem	13
2.3 Optimization of Second Equation of Kármán - Tsien by Ritz - Galerkin Method	15
2.3.1 Examples	15
 3.0 MATHEMATICAL FORMULATION OF THE PROBLEM - RESPONSE OF THE BUCKLED PLATE UNDER UNIAXIAL COMPRESSION AND LATERAL ACOUSTIC LOADING	 17
3.1 Method of Equivalent (Anisotropic) Orthotropic (Buckled) Plates	17

TABLE OF CONTENTS (Continued)

	Page
3.2 Dynamic Equations of Motion	17
3.3 Boundary and Initial Conditions	18
3.4 Mathematical Analysis	18
3.4.1 Homogeneous Solution	19
3.4.2 The Complete Solution	20
3.5 The Response of the Buckled Panel under Acoustic Loading	22
3.6 Evaluation of Elastic Constants of Equivalent Uniform Orthotropic Plates	23
4.0 CONCLUSIONS AND RECOMMENDATIONS	24
REFERENCES	25
ACKNOWLEDGEMENTS	26
APPENDIX A THE EQUATIONS OF EQUILIBRIUM OF A PLATE ELEMENT IN POST BUCKLED STATE	27
APPENDIX B THE DERIVATION OF EQUATION (8)	31
APPENDIX C APPLICATION OF FOURIER FINITE SINE TRANSFORMS WITH RESPECT TO X ON EQUATION (9)	32
APPENDIX D FOURIER FINITE SINE TRANSFORMS WITH RESPECT TO X OF $\cos^2 \frac{4\pi x}{a}$ AND $\sin^2 \frac{4\pi x}{a}$	36
APPENDIX E SOLUTION OF HOMOGENEOUS EQUATION $(D^4 - 2\beta^2 D^2 + \beta^4) \bar{F}_1(p, y) = 0$	37
APPENDIX F EVALUATION OF CONSTANTS E_k of (39)	38
APPENDIX G OPTIMIZATION OF (43) BY RITZ-GALERKIN METHOD	40

LIST OF FIGURES

Number	Title	Page
Figure 1	The specimen under study and its coordinate system .	46
Figure 2	The problem to be analyzed in this study .	46
Figure 3	A plate element of unit width under lateral and in-plane loadings .	46
Figure 4	The end view of a plate element under an axial end load P and lateral acoustic load $q_1 + q_2$.	47
Figure 5	The system and the associated response of Figure 5 are equivalent to those of Figure 4.	47
Figure 6	The buckled plate.	48
Figure 7	Non-dimensional compressive load versus non-dimensional buckle depth .	49

LIST OF TABLES

	Page
TABLE I NON-DIMENSIONAL BUCKLE DEPTH	45

NOTATION

D	flexural rigidity of plate
w	plate lateral deflection
x, y	coordinates in the plane of the plate
ν	Poisson's ratio
P	axial end load of "beam"
M	bending moment due to lateral load
h	plate thickness
F, \bar{F}	Airy stress functions
E	Young's modulus of elasticity
Δ, ∇^2	Laplacian operator
∇^4	biharmonic operator
N_{xi}	constant initial stress resultant
q	load per unit area
a	length of plate
b	width of plate
A_{41}	maximum depth of buckle
p	transform parameter
β	$\frac{p\pi}{a}$
D	$\frac{\partial}{\partial y}$
\bar{F}	inverse Fourier transform
k	$\frac{\text{total load on the panel}}{\text{Euler critical load of the panel}}$
ϵ	normal strain
γ	shear strain
σ	normal stress
τ	shear stress
u	displacement parallel to x axis
v	displacement parallel to y axis

NOTATION (Continued)

ρ	density
D_x, D_y, H	elastic constants of the anisotropic plate

1.0 INTRODUCTION

Aircraft and aerospace structures, because of severe weight limitations, normally use thin plates as structural elements. As a consequence, many structures are used in a post-buckled state. "The critical load for a rod is practically the ultimate load; the plate fixed on all edges, however, can support a load which exceeds the critical load tenfold" according to Bolotin (ref. 1), a well-known authority in the field of elastic stability.

In addition, thin plates subjected to compressive forces in one direction are expected to survive the environment of dynamic forces. The purpose of this study is to determine the dynamic characteristics and the structural response of the buckled panel.

The compressive loads arise from the inertia and thrust loadings and vehicle bending during adjustments in flight path. Also present, but not considered in this study, are the tension loads due to internal pressure and vehicle bending. In essence, it is estimated that many combinations of tension and compression can and do exist. The dynamic forces consist of oscillating shocks, steady and unsteady aerodynamic forces, and acoustic loads.

The present study is concerned with small vibrations of a rectangular plate, under a uni-axial inplane compressive load system, (as it is most critical) and lateral dynamic forces. The object of the present investigation is to obtain information regarding the response (deflection and stresses), frequencies, and mode shapes.

1.1 Literature Survey

A literature search of publications, in fields which are closely related to the problem at hand, is made. A representative list consists of the following references (refs. 2 - 6). The starting point, invariably, is the non-linear partial differential equations of large deflection theory of plates, known as von Kármán, Tsien equations (ref. 7). With few exceptions, most of the investigators utilized energy methods, perturbation methods, and power series methods (ref. 8) of analysis. The linear theory of plates (the deflections are small as compared to thickness) is not valid, if the static behavior of plates in post-buckled state is to be investigated. For the dynamic behavior of plates, the classical small vibration theory is valid as the amplitudes are "small"; and "jump phenomena" is not noticeable, under acoustic loads.

1.2 Method of Approach

In this study, it is proposed to use methods of analysis, which are conceptually different from those of the earlier investigators. The given problem is divided into two parts and the responses calculated separately; and then superposed.

The rectangular panel under study along with its coordinate system is shown in Figure 1. The problem, in Figure 1, is equivalent to that shown in Figure 2, if the regions at the ends in close proximity, (where the inplane loads of Figure 1 are replaced by the statically equivalent loads of Figure 2) are ignored.

Let us concentrate our attention on a plate element of width unity, as shown in Figures 3 and 4.

The present state of knowledge of the buckling strength of metallic structures can be summarized as follows. (ref. 9).

"The buckling load of a plate is considerably increased by lateral normal pressure. The normal pressure causes a much smaller increase in buckling load of a plate with fixed edges than that of a plate with simply supported edges".

The above comments by Bleich (ref. 9), based on the work of Levy and his associates stresses the importance of boundary conditions. It is very well recognized that boundary conditions play a significant role in static and dynamic behavior of structural elements (ref. 10).

A plate continuous over a number of supports and carrying normal and edgewise loads has edge conditions which are "partially fixed". However, in the technical literature, it is often found that mathematical convenience overruled the physical reality.

For simply supported plates, whose aspect ratio is 4, Levy and his associates assumed the number of half waves in lengthwise direction equals 5 (compressive load is parallel to the length), while Gerard (ref. 10) claims, it is 4. Levy also concluded that the assumption of 4 or 5 half waves introduces an error less than 5 percent in the final solution.

1.2.1 Superposition in a Modified Form of Beam-column Loadings and Responses

It is possible to superpose the response of the two beam-column effects of Figure 5, under certain conditions.

The deflections and bending moments (stresses) in beam-columns are not proportional to the axial end compressive loads. However, if the end compressive load remains constant, the responses can be superposed, (ref. 11) in a modified form.

For the simply supported "beam" of Figure 5 (a)

$$D \left[\frac{\partial^2 w_1}{\partial x^2} + \nu \frac{\partial^2 w_1}{\partial y^2} \right] = M_1 - P w_1 \quad (1)$$

where

M_1 is the bending moment due to the lateral load, D is the flexural rigidity of the

plate, ν is Poisson's ratio, w is lateral deflection, and P is the axial end compressive load.

For the "beam" of Figure 5 (b)

$$D \left[\frac{\partial^2 w_2}{\partial x^2} + \nu \frac{\partial^2 w_2}{\partial y^2} \right] = M_2 - P w_2 \quad (2)$$

where

M_2 is the bending moment due to the lateral load.

Adding (1) and (2), (3) is obtained:

$$\begin{aligned} D \left[\frac{\partial^2}{\partial x^2} (w_1 + w_2) + \nu \frac{\partial^2}{\partial y^2} (w_1 + w_2) \right] \\ = (M_1 + M_2) - P w_2 \end{aligned} \quad (3)$$

Equation (3) is the differential equation of bending of the "beam" of Figure 4, where $M_1 + M_2$ is the bending moment due to the total lateral loads $q_1 + q_2$, whose response is $w_1 + w_2$. Equation (3) is still valid in the special case when q_1 and M_1 are zero.

The only assumption made in the derivation of the superposition of the beam column effects is that the bending moment of a plate element is defined by the expression

$$D \left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right]$$

even in the post-buckled state. This analysis is performed under the restrictive assumption that the deflections of the plate are considered to be finite, even though sufficiently small, so that the slopes of the plate elements are negligible as compared to unity. The above assumption is realistic as many investigators, including Timoshenko (ref. 12) studied post-buckling behavior of plates, under the above limitations.

In Section 2.0, the deflection of a plate without any lateral load, but under a uni-axial compressive load, in a post-buckled state, is examined. In Section 3.0, the deflection of a buckled (corrugated) plate with lateral load and under a uniaxial compressive load, is analyzed.

The analysis is based on the assumption of elastic behavior of the plate.

2.0

MATHEMATICAL FORMULATION OF THE PROBLEM-STATIC DEFLECTION OF THE BUCKLED PLATE

In the following analysis, let the xoy plane be the middle surface of the plate, and let oz be the direction of the lateral deflection. The plate is subjected to in-plane static compressive loads, and to lateral acoustic loads in the z direction. A typical repeating section of the plate bounded by two stiffeners at two sides of the plate, is analyzed. The stiffeners are parallel to x axis as shown in Figure 2. The thickness of the plate is h . In the absence of body forces, the two well-known non-linear partial differential equations of compatibility and lateral equilibrium for bending of plates in the case of finite (but not too large) deflections known as von Kármán - Tsien equations (ref. 7) are (the deflections are no longer small as compared to thickness).

$$\Delta \Delta F = \nabla^4 F = E \left[w_{xy}^2 - w_{xx} w_{yy} \right] \quad (4)$$

$$q = 0 = D \nabla^4 w - h \left[F_{xx} w_{yy} + F_{yy} w_{xx} - 2F_{xy} w_{xy} + \frac{N_{xi}}{h} w_{xx} \right] \quad (5)$$

where E is the Young's Modulus of elasticity, $\Delta (= \nabla^2)$ represents the Laplacian Operator, F is the Airy stress function, w is the lateral deflection of the plate, N_{xi} is the constant initial stress resultant, h is the plate thickness, and q is the load per unit area applied to the lateral surface of the plate. The derivations of (4) and (5) are shown in Appendix A. The flexural rigidity is

$$D = \frac{E h^3}{12(1 - \nu^2)} \quad (6)$$

where ν is Poisson's ratio. Equations (4) and (5) are derived under the assumption that the deflections are finite, even though sufficiently small, so as to neglect the angles of rotation in comparison to unity. A consequence of this assumption is that the simplified formulae for curvatures of a plate are applicable. In this report, an approximate engineering solution is presented.

2.1 Method of Solution of Buckled Plate in Postbuckled State

The deflection $w(x, y)$ of the plate (Figure 6) in postbuckled state, is expressed as a series of assumed functions in the following form, after Levy and Navier (ref. 12)

$$w(x, y) = \sum_{m=4}^{\infty} \sum_{n=1,3}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \quad (7)$$

Since the plate is continuous in both x and y direction and the plate is connected at few isolated rivets to the stiffeners, in this report, simply supported edge conditions are assumed for the plate.

$w(x, y)$ should satisfy the assumed geometric boundary conditions; they need not satisfy the natural boundary conditions. The number of buckles in the x and y directions depend upon the plate geometry and boundary conditions (ref. 10). For the conditions assumed, m equals 4 and n equals 1, where m and n are the number of half waves in the x and y direction. However, note that explosive changes of wave pattern to a different stable position can and do occur. This phenomena is known as "oil-canning". Using (7) in (4), one can obtain F as shown below in the next section.

2.2 Calculation of Stress Function

2.2.1 First Approximation:

Replacing $w(x, y)$ by the first term of the infinite series; and substituting the derivatives of w in (4), one obtains (refer to Appendix B for details.)

$$\nabla^4 F = E A_{41}^2 \left(\frac{4\pi}{a} \right)^2 \left(\frac{\pi}{b} \right)^2 \left[\cos^2 \frac{4\pi x}{a} \sin^2 \frac{\pi y}{b} - \sin^2 \frac{4\pi x}{a} \cos^2 \frac{\pi y}{b} \right] \quad (8)$$

$$\nabla^4 F = B_1 \left[f_1(y) \cos^2 \frac{4\pi x}{a} - f_2(y) \sin^2 \frac{4\pi x}{a} \right] \quad (9)$$

where

$$B_1 = E A_{41}^2 \left(\frac{4\pi}{a} \right)^2 \left(\frac{\pi}{b} \right)^2$$

$$f_1 = \sin^2 \frac{\pi y}{b}$$

and

$$f_2 = \cos^2 \frac{\pi y}{b} .$$

2.2.2 Fourier Finite Sine Transform of the Nonhomogeneous Partial Differential Equation.

There are various methods by which (9) can be solved. One convenient method of analysis, (refs. 13 and 14) is by application of Fourier finite sine transforms with respect to x on (9). The boundary conditions on F at the simply supported edges $x = 0$ and $x = a$ are (ref. 6)

$$F = F_{xx} = 0. \quad (10)$$

The Fourier finite sine transform with respect to x is defined by the following

$$T[F(x, y)] = \bar{F}(p, y) = \int_0^a F(x, y) \sin \frac{p\pi x}{a} dx. \quad (11)$$

Utilizing (10), (11), and Appendix C, (9) reduces to

$$\begin{aligned} [D^4 - 2\beta^2 D^2 + \beta^4] \bar{F}(p, y) &= B_1 f_1(y) L_1(a, p) \\ &+ B_1 f_2(y) L_2(a, p) \end{aligned} \quad (12)$$

where $D = \frac{\partial}{\partial y}$, and $\beta = \frac{p\pi}{a}$.

Further, L_1 and L_2 are the Fourier finite sine transforms with respect to x of $\cos^2 \frac{4\pi x}{a}$ and $\sin^2 \frac{4\pi x}{a}$ respectively. [The details of the derivation of L_k ($k = 1, 2$) are shown in Appendix D].

Note that

$$L_1 = \frac{a}{\pi(8+p)} \left[\left\{ -(-1)^p + 1 \right\} - 2 \left\{ \frac{2}{p} [(-1)^p - 1] + \frac{(-1)^p - 1}{\left(\frac{p}{2} - 4\right)} \right\} \right] \quad (13)$$

and

$$L_2 = \frac{-2a}{\pi(8+p)} \left[(-1)^p - 1 \right] \left[\frac{2}{p} + \frac{1}{2\left(2 - \frac{p}{4}\right)} \right]. \quad (14)$$

2.2.3 Complete Formal Solution of Stress Function.

To solve (12), it is convenient to separate \bar{F} into two parts, the homogeneous solution \bar{F}_1 , and the particular integral \bar{F}_2 , where

$$\bar{F} = \bar{F}_1 + \bar{F}_2 \quad (15)$$

The solution of the homogeneous equation

$$\left(D^4 - 2\beta^2 D^2 + \beta^4 \right) \bar{F}_1(p, y) = 0 \quad (16)$$

is well-known and is (see Appendix E for details)

$$\begin{aligned} \bar{F}_1(p, y) = & C_1 \sinh \beta y + C_2 \cosh \beta y \\ & + C_3 y \sinh \beta y + C_4 y \cosh \beta y \end{aligned} \quad (17)$$

Particular Integral:

The method of undetermined coefficients is utilized to obtain the particular solution \bar{F}_2 of the differential equation (12).

$$\left(D^4 - 2\beta^2 D^2 + \beta^4 \right) \bar{F}_2(p, y) = B_1 f_1 L_1 + B_1 f_2 L_2 \quad (18)$$

Let

$$\bar{F}_2(p, y) = B_1 L_1 \bar{F}_{21}(p, y) + B_1 L_2 \bar{F}_{22}(p, y) \quad (19)$$

where \bar{F}_{21} and \bar{F}_{22} are solutions of equations, (20) and (21). Note superposition of solutions is valid for a linear differential equation.

$$\left(D^4 - 2\beta^2 D^2 + \beta^4 \right) \bar{F}_{21} = f_1 \quad (20)$$

and

$$\left(D^4 - 2\beta^2 D^2 + \beta^4 \right) \bar{F}_{22} = f_2 \quad (21)$$

$$\bar{F}_{21} = \frac{1}{2} \left[\frac{1}{\beta^4} - \frac{\sin \frac{2\pi}{b} y}{\left(\beta^2 + \frac{4\pi^2}{b^2} \right)^2} \right] \quad (22)$$

$$\bar{F}_{22} = \frac{1}{2} \left[\frac{1}{\beta^4} + \frac{\sin \frac{2\pi}{b} y}{\left(\beta^2 + \frac{4\pi^2}{b^2} \right)^2} \right] \quad (23)$$

Now (12) is rewritten as (24), using (15), (17), (18), (19), (22), and (23).

The complete solution of (12) is given by

$$\begin{aligned} \bar{F} = & C_1 \sinh \beta y + C_2 \cosh \beta y \\ & + C_3 y \sinh \beta y + C_4 y \cosh \beta y \\ & + B_1 L_1 \bar{F}_{21} + B_1 L_2 \bar{F}_{22} \end{aligned} \quad (24)$$

where

\bar{F}_{21} and \bar{F}_{22} are defined by (22) , (23) ;

L_1 and L_2 are defined by (13) , (14) .

In view of the symmetry of loading , geometry , and boundary conditions,

$$\bar{F}(p, \frac{b}{2}) = \bar{F}(p, -\frac{b}{2}) \quad (25)$$

$$\therefore C_1 = 0 \quad \text{and} \quad C_4 = 0 \quad (26)$$

At the unloaded, supported edges $y = \pm \frac{b}{2}$ (ref. 6) :

$$F = 0 \quad (27)$$

and

$$F_{yy} = 0 \quad (28)$$

Let us take Fourier finite sine transforms of the boundary conditions (30):

$$T \left[F_{yy} \left(x, \frac{b}{2} \right) \right] = T \left[F \left(x, \frac{b}{2} \right) \right] = 0 \quad (29)$$

$$T \left[F_{yy} \left(x, -\frac{b}{2} \right) \right] = T \left[F \left(x, -\frac{b}{2} \right) \right] = 0 \quad (30)$$

Therefore (29) and (30) reduce to

$$\bar{F}_{yy} \left(p, \frac{b}{2} \right) = \bar{F} \left(p, \frac{b}{2} \right) = 0 \quad (31)$$

$$\bar{F}_{yy} \left(p, -\frac{b}{2} \right) = \bar{F} \left(p, -\frac{b}{2} \right) = 0 \quad (32)$$

Substitution of (26) in (24) reduces (24) to (33), denoted by

$$\begin{aligned} \bar{F} = & C_2 \cosh \beta y + C_3 y \sinh \beta y \\ & + B_1 L_1 \bar{F}_{21} + B_1 L_2 \bar{F}_{22} \end{aligned} \quad (33)$$

where

B_1 is defined by (9) ,

L_1 and L_2 are defined by (13) and (14)

\bar{F}_{22} and \bar{F}_{22} are defined by (22) and (23) .

Applying the two transformed boundary conditions of (31) on (33), one derives

$$C_2 = \frac{-B_1 \left(\operatorname{sech} \frac{\beta b}{2} \right) (L_1 + L_2)}{4\beta^4} \left[\frac{\beta b}{2} \tanh \frac{\beta b}{2} + 2 \right] \quad (34)$$

and

$$C_3 = \frac{B_1 (L_1 + L_2)}{4\beta^3} \operatorname{sech} \frac{\beta b}{2} . \quad (35)$$

$$\begin{aligned} \bar{F}(p, y) = & C_2 \cosh \beta y + C_3 y \sinh \beta y \\ & + B_1 \left[L_1 \bar{F}_{21} + L_2 \bar{F}_{22} \right] \end{aligned} \quad (36)$$

where

C_2 and C_3 are defined by (34) and (35) .

The inverse Fourier Finite sine transform (ref. 13) with respect to x is defined by

$$F(x, y) = \frac{2}{a} \sum_{p=1}^{\infty} \bar{F}(p, y) \sin \frac{p\pi x}{a} \quad (37)$$

If we assume that the first term of the truncated infinite series is a good approximation (replace p by 1), then (37) reduces to (38)

$$F(x, y) \approx \frac{2}{a} \bar{F}(1, y) \sin \frac{\pi x}{a} \quad (38)$$

An approximate stress function is formally derived, as presented in (39).

$$\begin{aligned}
F(x, y) \approx \frac{2}{a} \left\{ \frac{-B_1 \left(\operatorname{sech} \frac{\beta b}{2} \right) (\bar{L}_1 + \bar{L}_2)}{4\beta^4} \left[\frac{\beta b}{2} \tanh \frac{\beta b}{2} + 2 \right] \cosh \beta y \right. \\
+ \left[\frac{B_1 (\bar{L}_1 + \bar{L}_2)}{4\beta^3} \operatorname{sech} \frac{\beta b}{2} \right] y \sinh \beta y \\
\left. + B_1 \left[\bar{L}_1 \bar{F}_{21} + \bar{L}_2 \bar{F}_{22} \right] \right\} \sin \frac{\pi x}{a} \quad (39)
\end{aligned}$$

where \bar{L}_1 and \bar{L}_2 define (13) and (14) with $p = 1$.

2.2.4 Application to a Specific Problem.

Let us apply the analysis to a specific problem, a typical panel of Saturn S II. (See Appendix F).

From (13) and (14)

$$\bar{L}_1 = \frac{62a}{63\pi} \quad (40)$$

$$\bar{L}_2 = \frac{64a}{63\pi} \quad (41)$$

$$\begin{aligned}
F(x, y) \approx \left[E_1 \cosh \beta y + E_2 y \sinh \beta y + (E_3 + E_5) \right. \\
\left. + (E_4 + E_6) \sin \frac{2\pi}{b} y \right] \sin \frac{\pi x}{a} \quad (42)
\end{aligned}$$

where

$$\begin{aligned}
 E_1 &= -5.545815093792 \times 10^8 A_{41}^2 \\
 E_2 &= 5.936356937513 \times 10^9 A_{41}^2 \\
 E_3 &= 7.217449305230 \times 10^{10} A_{41}^2 \\
 E_4 &= -1.708272024815 \times 10^7 A_{41}^2 \\
 E_5 &= 7.450270250789 \times 10^{10} A_{41}^2 \\
 E_6 &= 1.607793440693 \times 10^{-1} A_{41}^2 .
 \end{aligned}$$

The above values of E_k ($k = 1, 2, \dots, 6$) are obtained on a computer. [For details, refer to Appendix F.]

The derivatives of (42) are as follows:

$$\begin{aligned}
 F_{xx} \approx & -\left(\frac{\pi}{a}\right)^2 \left[E_1 \cosh \beta y + E_2 y \sinh \beta y + (E_3 + E_5) \right. \\
 & \left. + (E_4 + E_6) \sin \frac{2\pi}{b} y \right] \sin \frac{\pi x}{a}
 \end{aligned} \tag{43}$$

$$\begin{aligned}
 F_{xy} \approx & \left(\frac{\pi}{a}\right) \left[E_1 \beta \sinh \beta y + E_2 (y \beta \cosh \beta y + \sinh \beta y) \right. \\
 & \left. + \frac{2\pi}{b} (E_4 + E_6) \cos \frac{2\pi}{b} y \right] \cos \frac{\pi x}{a}
 \end{aligned} \tag{44}$$

$$\begin{aligned}
 F_{yy} \approx & \left[E_1 \beta^2 \cosh \beta y + E_2 \beta (y \beta \sinh \beta y + 2 \cosh \beta y) \right. \\
 & \left. - \left(\frac{2\pi}{b}\right)^2 (E_4 + E_6) \sin \frac{2\pi}{b} y \right] \sin \frac{\pi x}{a}
 \end{aligned} \tag{45}$$

2.3 Optimization of Second Equation of Kármán - Tsien by Ritz - Galerkin Method.

Now, w as defined by the first term of (7) and F , as defined by (42) and N_{xi} from ref. 9 are substituted in (5). Since w and the associated F are approximate, (5) does not vanish. However, the expression (5) is a cubic in the unknown parameter A_{41} . Therefore (5) should be satisfied in the mean, approximately, by utilizing the Ritz - Galerkin method, defined by (ref. 3)

$$\int_0^a \int_{-b/2}^{+b/2} q \frac{\partial w}{\partial A_{41}} dx dy = 0 \quad (46)$$

[The detailed evaluation of (46) is shown in Appendix G] .

Reference 15 is utilized in the evaluation of the integrals of Appendix G.

2.3.1 Examples

Let us concentrate our attention on two different thicknesses of plate, namely $h = 0.040''$ (Case 1) and $h = 0.032''$ (Case 2) . The final results of Cases 1 and 2 are presented in Table 1.

For Case 1,

$$h = 0.040''$$

Eq. (46) reduces to

$$A_{41} \left[(303.05 - 314.42 k) + (-107 + 437 - 5.44) 10^6 A_{41}^2 \right] = 0 \quad (47)$$

where

$$k = \frac{\text{total load on the panel}}{\text{Euler critical load of the panel}} .$$

$$A_{41} = 0 \quad (48)$$

is a trivial solution, corresponding to the unbuckled state of equilibrium condition.

If $k = 1$, i.e., at the onset of buckling A_{41} equals 0.000175 , and

$$\frac{w_{\max}}{h} \text{ equals } 0.0044 .$$

For Case 2 ,

$$h = 0.032''$$

Eq. (46) reduces to

$$A_{41} \left[(155.16 - 160.98 k) + (-85.7 + 349.7 - 4.35) 10^6 A_{41}^2 \right] = 0. \quad (49)$$

From reference 16, we know that a plate with 4 bays was subjected to 58,540 lbs. Therefore, if we assume that each bay takes 1/4 of 58,540 lbs. or 14,635 lbs., then

$$k = \frac{N_x}{N_{x \text{ cr}}} = \frac{14,635}{273.6} \approx 54. \quad (50)$$

The value of $N_{x \text{ cr}}$ is obtained from reference 9.

For two different values of h , A_{41} which defines w_{max} is calculated. A non-dimensional curve, defining k versus $\frac{w_{\text{max}}}{h}$ is plotted in Fig. 7., and the values corresponding to the nondimensional curve are evaluated in Table 1. It is observed that all the values fall on the same nondimensional curve. If the non-linear terms in (47) and (48) are ignored as a first approximation and if $k = 1$, the equations are satisfied within engineering accuracy. These two observations suggest that the analysis is reasonably free from algebraic errors. However, the author feels that the values for the depth of the buckle look low, probably due to the fact that infinite series were replaced by single terms.

3.0 MATHEMATICAL FORMULATION OF THE PROBLEM - RESPONSE OF THE BUCKLED PLATE UNDER UNIAXIAL COMPRESSION AND LATERAL ACOUSTIC LOADING

As explained earlier in Section 1.2.1, the second component of the response, which is dynamic in nature, can be calculated by one of the following methods.

- (1) strain energy methods
- (2) methods of shallow shell theory
- (3) plate with initial imperfections
- (4) anisotropic plate theory

For the present analysis, it is proposed to apply the anisotropic plate theory type of analysis, as it is simple; consistent with the fact that it gives reasonably accurate answers (ref. 17) in a limited time.

3.1 Method of Equivalent (Anisotropic) Orthotropic (Buckled) Plates.

For ease of handling the problem analytically, the buckled plate is replaced by a fictitious orthotropic plate of uniform thickness. (Materials which have three mutually orthogonal planes of elastic symmetry are said to be orthotropic.) The equivalent elastic constants of the orthotropic plates should be obtained from the buckled state. The elastic constants are used in the linear non-homogeneous partial differential equation shown in equation (51). The orthotropic theory and the method of analysis are not new and are widely used in the calculation of static and dynamic responses of corrugated plates, reinforced concrete slabs, laminated structural panels, wooden plates etc.

3.2 Dynamic Equations of Motion

Let $w(x, y, t)$ be the transverse displacement of a vibrating orthotropic plate of uniform thickness h . The governing equation for the deflection w , under the combined action of the acoustic loading q and edgewise compression force N_x (neglecting the rotary inertia and shear deformation) is:

$$D_x \frac{\partial^4 w}{\partial x^4} + 2 \left(D_1 + 2 D_{xy} \right) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - N_{xi} \frac{\partial^2 w}{\partial x^2} = q(x, y, t) - \rho h \frac{\partial^2 w}{\partial t^2} \quad (51)$$

Assume the plate is subjected to a normal uniform (constant) acoustic loading.

Introducing the notation,

$$H = D_1 + 2 D_{xy} \quad (52)$$

Equation (2) reduces to:

$$\left. \begin{aligned} D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - N_{xi} \frac{\partial^2 w}{\partial x^2} \end{aligned} \right\} = q - \rho h \frac{\partial^2 w}{\partial t^2} \quad (53)$$

The solution of (53) with the associated boundary values and initial conditions is the response of the buckled plate under (dynamic) acoustic loads.

3.3 Boundary and Initial Conditions

Referring to Figure 6, the boundary conditions at all the four edges are assumed to be simply supported. Initial conditions for the fictitious corrugated plate may be assumed to be specified displacement and velocity at $t = 0$.

3.4 Mathematical Analysis

Equation (53) is a linear, partial, non-homogeneous differential equation. There are various classical methods available in the literature for its solution that are consistent with the assumed boundary and initial conditions. We follow the "variable separable" method, also called the product solution.

We assume that

$$w(x, y, t) = X(x) Y(y) T(t) \quad (54)$$

In view of the geometric boundary conditions at the edges $x = 0$, $x = a$, and $y = \pm \frac{b}{2}$, the deflection surface function (54) may be written as

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sin \frac{m\pi x}{a} \cos \frac{(2n+1)\pi y}{b} T_{mn}(t) \quad (55)$$

3.4.1 Homogeneous Solution

Substituting (55) in (53), with $q = 0$ and dividing the entire expression by

$\sin \frac{m\pi x}{a} \cos \frac{(2n+1)\pi y}{b} T_{mn}$, one obtains :

$$\left[D_x \left(\frac{m\pi}{a} \right)^4 + 2H \left(\frac{m\pi}{a} \right)^2 \left(\frac{2n+1}{b} \right)^2 \pi^2 + D_y \left(\frac{2n+1}{b} \right)^4 \pi^4 + N_{xi} \left(\frac{m\pi}{a} \right)^2 + \rho h \frac{\ddot{T}_{mn}}{T_{mn}} \right] = 0 \quad (56)$$

where dot denotes differentiation with respect to independent variable, time t .

In the following analysis, we focus our attention on a single harmonic and delete the subscripts mn , up to end of (61) separating the space and time variables of (56), we have

$$D_x \left(\frac{m\pi}{a} \right)^4 + 2H \left(\frac{m\pi}{a} \right)^2 \left(\frac{2n+1}{b} \right)^2 \pi^2 + D_y \left(\frac{2n+1}{b} \right)^4 \pi^4 + N_{xi} \left(\frac{m\pi}{a} \right)^2 = - \rho h \frac{\ddot{T}_{mn}}{T_{mn}} = g, \text{ a constant} \quad (57)$$

where g is real (positive, negative or zero) or complex.

$$\therefore - \rho h \ddot{T}_{mn} = g T_{mn} \quad (58)$$

$$\text{i.e., } \ddot{T}_{mn} = - g_1 T_{mn} \quad (59)$$

$$\text{where } g_1 = - \frac{g}{\rho h}$$

and g_1 is real (positive or negative) or complex. The possibility that $g_1 = 0$ is ruled out as $T(t)$ should be bounded for large values of t .

Solution of (59) reduces to

$$T = G_1 e^{\sqrt{g_1} t} + G_2 e^{-\sqrt{g_1} t} . \quad (60)$$

If g_1 is real and negative, $T(t)$ has bounded solutions; if g_1 is real and positive, T has unbounded solutions and dynamic instability sets in. For this analysis, we assume that there is no instability and the solutions are bounded.

$$\therefore g_1 \text{ is negative and let } g_2 = -g_1 \text{ and } g_2 > 0.$$

Equation (60) may be rewritten as follows:

$$T = G_3 \cos \sqrt{g_2} t + G_4 \sin \sqrt{g_2} t \quad (61)$$

where G_3 and G_4 are constants to be evaluated from the initial conditions.

Equation (55) reduces to

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sin \frac{m\pi x}{a} \cos \frac{(2n+1)\pi y}{b} \left[G_3 \cos \sqrt{g_{2_{mn}}} t + G_4 \sin \sqrt{g_{2_{mn}}} t \right] . \quad (62)$$

3.4.2 The Complete Solution

Let us assume that the acoustic load may be expressed as:

$$q(x, y, t) = Q_0(x, y) f(t) . \quad (63)$$

Assume that it is possible to expand the spatial distribution of load $Q_0(x, y)$, as an infinite series, similar to the expansion of (62). Then

$$Q_0(x, y) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} q_{mn} \sin \frac{m\pi x}{a} \cos \frac{(2n+1)\pi y}{b} \quad (64)$$

where

$$q_{mn} = \frac{\int_0^a \int_{-\frac{b}{2}}^{+\frac{b}{2}} Q_0(x, y) \sin \frac{m\pi x}{a} \cos \frac{(2n+1)\pi y}{b} dx dy}{\int_0^a \int_{-\frac{b}{2}}^{+\frac{b}{2}} \left[\sin \frac{m\pi x}{a} \cos \frac{(2n+1)\pi y}{b} \right]^2 dx dy} \quad (65)$$

Note

$$\begin{aligned} \int_0^a \int_{-\frac{b}{2}}^{+\frac{b}{2}} \left[\sin \frac{m\pi x}{a} \cos \frac{(2n+1)\pi y}{b} \right] \left[\sin \frac{p\pi x}{a} \cos \frac{(2q+1)\pi y}{b} \right] dx dy \\ = 0 \quad \text{if } m \neq p \quad \text{or } n \neq q \end{aligned} \quad (66)$$

as the orthogonality property between the mode shapes still holds good.

The assumed solution for w namely (55), and (61) are substituted in (57) and we obtain the natural frequency as

$$g_{2mn} = \frac{1}{\sqrt{\rho h}} \left[D_x \left(\frac{m\pi}{a} \right)^4 + 2H \left(\frac{m}{a} \right)^2 \left(\frac{2n+1}{b} \right)^2 \pi^4 + D_y \left(\frac{2n+1}{b} \right)^4 \pi^4 + N_x \left(\frac{m\pi}{a} \right)^2 \right]^{1/2} \quad (67)$$

The acoustic load is a time varying force, of an arbitrary nature, the time dependent part of $w(x, y, t)$ should include an additional contribution of Duhamel integral type of representation, defined by

$$\frac{q_{mn}}{\sqrt{g_{2mn}}} \int_0^t f(\tau) \sin \omega(t - \tau) d\tau$$

where

q_{mn} is defined by (65) and

g_{2mn} by (67) .

Now (62) reduces to

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left[\sin \frac{m\pi x}{a} \cos \frac{(2n+1)\pi y}{b} \right] \left[A \cos \sqrt{g_{2mn}} t + B \sin \sqrt{g_{2mn}} t + \frac{q_{mn}}{g_{2mn}} \int_0^t f(\tau) \sin \omega(t - \tau) d\tau \right] \quad (68)$$

A formal solution of the dynamic response of a buckled plate is presented. To evaluate (68) explicitly, knowledge of A , B , and $f(t)$ are required.

3.5

The Response of the Buckled Panel under Acoustic Loading

The deflection of the panel consists of two components, namely the static component and the dynamic component. The static deflection component is defined approximately by (7) with one term, where the coefficient A_{41} is defined by the appropriate value for the plate of specified thickness. The dynamic component is defined by (68) for any specified initial conditions and acoustic loading. Explicit numerical values for the maximum principal strains is not possible at this stage without a specific definition of the acoustic loading. However, the analysis can be made as follows:

$$w(x, y, t) = w_1(x, y) + w_2(x, y, t) \quad (69)$$

(Eq. 7) (Eq. 68)

Having obtained w , following Timoshenko's notation (ref. 12), the following generalized forces can be derived.

$$M_x = - D_x \frac{\partial^2 w}{\partial x^2} - D_1 \frac{\partial^2 w}{\partial y^2} \quad (70)$$

$$M_y = - D_1 \frac{\partial^2 w}{\partial x^2} - D_y \frac{\partial^2 w}{\partial y^2} \quad (71)$$

$$M_{xy} = - 2D_{xy} \frac{\partial^2 w}{\partial x \partial y} \quad (72)$$

Now the bending stresses can be expressed in terms of the bending moments as follows:

$$\begin{aligned} \sigma_x &= \frac{12 M_x z}{h^3} \\ \tau_{xy} &= \frac{12 M_{xy} z}{(1 - \nu^2) h^3} \\ \sigma_y &= \frac{12 M_y z}{h^3} \end{aligned} \quad (73)$$

The maximum bending stresses occur at $z = \pm \frac{h}{2}$.

From the stresses defined by (23) one can derive the principal stresses by using Mohr's circle or otherwise; and the principal strains by using (A1).

3.6 Evaluation of Elastic Constants of Equivalent Uniform Orthotropic Plates

$$\begin{aligned} h &= 0.040'' \\ \text{assume } l &\approx s \\ D_x &= D_y = 63 \\ H &= 42 \\ D_1 &\approx 0 \end{aligned}$$

[The semi-empirical formulae presented by Seydel and reproduced on p. 367 of (12) are utilized.]

The present theoretical analysis is based under certain assumptions mentioned below: The four edges are simply supported. The modified stress function includes the end compressive forces, which are constant even in post buckled state. The double infinite series for w is replaced by the first term of the series. Similarly, the inverse of the stress function, also an infinite series, is approximated by its first term. The second nonlinear partial differential equation of von Kármán is solved in an average manner. All the above simplifying assumptions do introduce certain errors. The methodology applied is conceptually different from any available in the literature. A very interesting feature of the analysis is Figure 7. From the basic set of equations, the depth of the buckle for two identical sheets, but of different thicknesses, is calculated for various end loads in post buckled region. When plotted in a non-dimensional manner, both the curves reduce to one master curve, as they should.

The boundary conditions play a major role in all buckling problems; as such, the realistic boundary conditions should be introduced in the theoretical analysis. If one replaces any infinite series, by finite number of terms, one should consider more than one term. However, this suggestion increases the algebra considerably.

To calculate the dynamic response, the buckled sheet is replaced by a fictitious uniformly thick anisotropic plate. The buckled sheet which has 4 half waves in one direction and 1 half wave in a perpendicular direction is replaced by a corrugated sheet, with corrugations in one direction only. This is a crude approximation which probably led to a great sacrifice in accuracy. For a more precise analysis, a more realistic, equivalent anisotropic plate analysis has to be used.

Equation (68) defines the dynamic response of a buckled sheet, under acoustic loading. For various mathematically tractable forcing functions, (68) can be evaluated.

The value of this report could be enhanced by use of electronic computers and additional analysis.

REFERENCES

1. Bolotin, V.V. (Weingarten, V.I., et.al. trans.): The Dynamic Stability of Elastic Systems. Holden-Day, Inc., 1964, p. 399.
2. Chang, Y.W. : Vibrations and Stability of Buckled Rectangular Plates. Ph. D. Thesis, University of Michigan, 1964.
3. Van der Neut, A. : Post Buckling Behaviour of Structures. AGARD Report 60, NATO Advisory Group for Aeronautical Research and Development, 1956.
4. Herzog, B ; and Masur, E.F. : Frequencies and Modes of Vibration of Buckled Circular Plates, NASA TN D - 2245, 1964.
5. Levy, S. : Large Deflection Theory for Rectangular Plates. Proceedings of Symposia in Applied Mathematics. Am. Math. Society, New York, Vol. 1, 1949.
6. Cummings, B.E. : Some Nonlinear Vibration and Response Problems of Cylindrical Panels and Shells. GALCIT Report, SM 62-32, June 1962.
7. Von Kármán, Th. : Festigkeitsprobleme im Maschinenbau. Enzyklopädie der Mathematischen Wissenschaften, Vol. 4, 1910, Article 27.8, p. 348.
8. Mansfield, E.H. : The Bending and Stretching of Plates. Pergamon Press Book. Macmillan Co., New York 1964, pp. 114-143.
9. Bleich, F. : Buckling Strength of Metal Structures. McGraw-Hill Book Co., New York, 1952, pp. 494-498.
10. Gerard, G. : Introduction to Structural Stability Theory. McGraw-Hill Book Co., New York, 1962, pp. 54-70.
11. Timoshenko, S.P. ; and Gere, J.M. : Theory of Elastic Stability. McGraw-Hill Book Co., Inc., 1961, p. 7 and p. 346.
12. Timoshenko, S. ; and Woinowsky-Krieger, S: Theory of Plates and Shells. McGraw-Hill Book Co., Inc., 1959, p. 108.
13. Sneddon, I.N. : Fourier Transforms. McGraw-Hill Book Co., Inc., 1951.
14. Amba-Rao, C.L. : On the Vibration of a Rectangular Plate Carrying a Concentrated Mass. J. Appl. Mech. Trans. ASME, Vol. 3, 1964, pp. 550-551.
15. Gradshteyn, I.S. ; and Ryzhik, I. : Tables of Integrals, Series and Products Academic Press, New York, 1965.
16. Berry, J. : Wyle Laboratories Research Staff - TM 66-7 ; and TM 66-8, 1966.

ACKNOWLEDGMENTS

The author is highly indebted to Mr. A. Wenzel for valuable technical discussions. Thanks are due to Messrs. B. Spice and J. Berry for drawing the author's attention to some minor details of earlier experimental work; and to Mr. J. Cockburn for verifying some integrals.

APPENDIX A
THE EQUATIONS OF EQUILIBRIUM OF A PLATE ELEMENT
IN POST BUCKLED STATE

Let u , v , and w be the displacements parallel to the axes of x , y , and z , respectively, where the z axis is normal to the plate; the equations of equilibrium of a plate element can be derived from the well-known static equations of an element, treated as a free body.

Assuming σ_z is negligible, the stress strain relations for a thin plate element in bending are

$$\begin{aligned}\epsilon_x &= \frac{1}{E} (\sigma_x - \nu \sigma_y) \\ \epsilon_y &= \frac{1}{E} (\sigma_y - \nu \sigma_x) \\ \gamma_{xy} &= \frac{1}{G} (\tau_{xy})\end{aligned}\tag{A1}$$

where

$$G = \frac{E}{2(1 + \nu)}.$$

From (A1), we obtain

$$\begin{aligned}\sigma_x &= \frac{E}{1 - \nu^2} (\epsilon_x + \nu \epsilon_y) \\ \sigma_y &= \frac{E}{1 - \nu^2} (\epsilon_y + \nu \epsilon_x) \\ \tau_{xy} &= G \gamma_{xy} = \frac{E}{2(1 + \nu)} \gamma_{xy}\end{aligned}\tag{A2}$$

If there are no body forces acting in the middle plane of the plate, the differential equations of equilibrium of an element $dx dy$ in the plane of the element become

$$\frac{Eh}{1-\nu^2} \left[\frac{\partial \epsilon_x}{\partial x} + \nu \frac{\partial \epsilon_y}{\partial x} + \frac{1}{2} (1-\nu) \frac{\partial \gamma_{xy}}{\partial y} \right] = 0 \quad (A6)$$

$$\frac{Eh}{1-\nu^2} \left[\frac{\partial \epsilon_y}{\partial y} + \nu \frac{\partial \epsilon_x}{\partial y} + \frac{1}{2} (1-\nu) \frac{\partial \gamma_{xy}}{\partial x} \right] = 0$$

With in-plane and normal loadings, q (lb per sq. in.), the equation of equilibrium in z direction reduces to

$$D \nabla^4 w = \frac{Eh}{1-\nu^2} \left[(\epsilon_x + \nu \epsilon_y) \frac{\partial^2 w}{\partial x^2} + (\epsilon_y + \nu \epsilon_x) \frac{\partial^2 w}{\partial y^2} + (1-\nu) \gamma_{xy} \frac{\partial^2 w}{\partial x \partial y} \right] + q(x,y) \quad (A7)$$

where
$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

and
$$D = \frac{Eh^3}{12(1-\nu^2)}$$

Following Timoshenko and von Kármán, it is assumed that the deflections are not small as compared to plate thickness, but small enough to justify the use of the application of simplified formulae for curvature of a plate element.

The strain displacement relations for the large deflection plate theory are:

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ \epsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{aligned} \quad (A8)$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (A3)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

where N_x , N_y , and N_{xy} are normal forces per unit length parallel to x and y axes and shear force in xy plane (dimensions $\frac{lb}{in.}$) respectively.

A great simplification can be made if one introduces a stress function defined by \bar{F} , which satisfies (A3).

$$\begin{aligned} N_x &= h \frac{\partial^2 \bar{F}}{\partial y^2} \\ N_y &= h \frac{\partial^2 \bar{F}}{\partial x^2} \\ N_{xy} &= -h \frac{\partial^2 \bar{F}}{\partial x \partial y} \end{aligned} \quad (A4)$$

Noting that

$$\begin{aligned} N_x &= \sigma_x h \\ N_y &= \sigma_y h \\ N_{xy} &= \tau_{xy} h \end{aligned} \quad (A5)$$

and substituting (A2) in (A3), one derives the following equations.

Differentiating the two equations of (A6) by x and y , respectively, and adding the two equations, utilizing (A4) and (A8); (A6) can be replaced by the compatibility equation defined by

$$\nabla^4 \bar{F} - E \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] = 0 \quad (A9)$$

Utilizing (A4) and (A8), (A7) reduces to

$$q = D \nabla^4 w - h \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \bar{F}}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 \bar{F}}{\partial x^2} - 2 \frac{\partial^2 \bar{F}}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right] \quad (A10)$$

If the plate is under uniform uniaxial loading as in Figure 2, it is convenient to separate the initial stress resultant N_{xi} from N_x and define a new stress function, defined by

$$F = \bar{F} - \frac{1}{2} y^2 \frac{N_{xi}}{h} \quad (A11)$$

Then (A9) and (A10) are replaced by (A12) and (A13) defined by

$$\nabla^4 F - E \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] = 0 \quad (A12)$$

and

$$q = D \nabla^4 w - h \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 F}{\partial x^2} + \frac{N_{xi}}{h} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right] \quad (A13)$$

APPENDIX B

THE DERIVATION OF EQUATION (8)

Let us work with one term of the infinite series of (7) .

Let

$$w = A_{41} \sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} . \quad (B1)$$

Then

$$\frac{\partial^2 w}{\partial x \partial y} = -A_{41} \left(\frac{4\pi}{a} \right) \left(\frac{\pi}{b} \right) \cos \frac{4\pi x}{a} \sin \frac{\pi y}{b} \quad (B2)$$

$$\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 = A_{41}^2 \left(\frac{4\pi}{a} \right)^2 \left(\frac{\pi}{b} \right)^2 \cos^2 \frac{4\pi x}{a} \sin^2 \frac{\pi y}{b} \quad (B3)$$

$$\frac{\partial^2 w}{\partial x^2} = -A_{41} \left(\frac{4\pi}{a} \right)^2 \sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} \quad (B4)$$

$$\frac{\partial^2 w}{\partial y^2} = -A_{41} \left(\frac{\pi}{b} \right)^2 \sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} . \quad (B5)$$

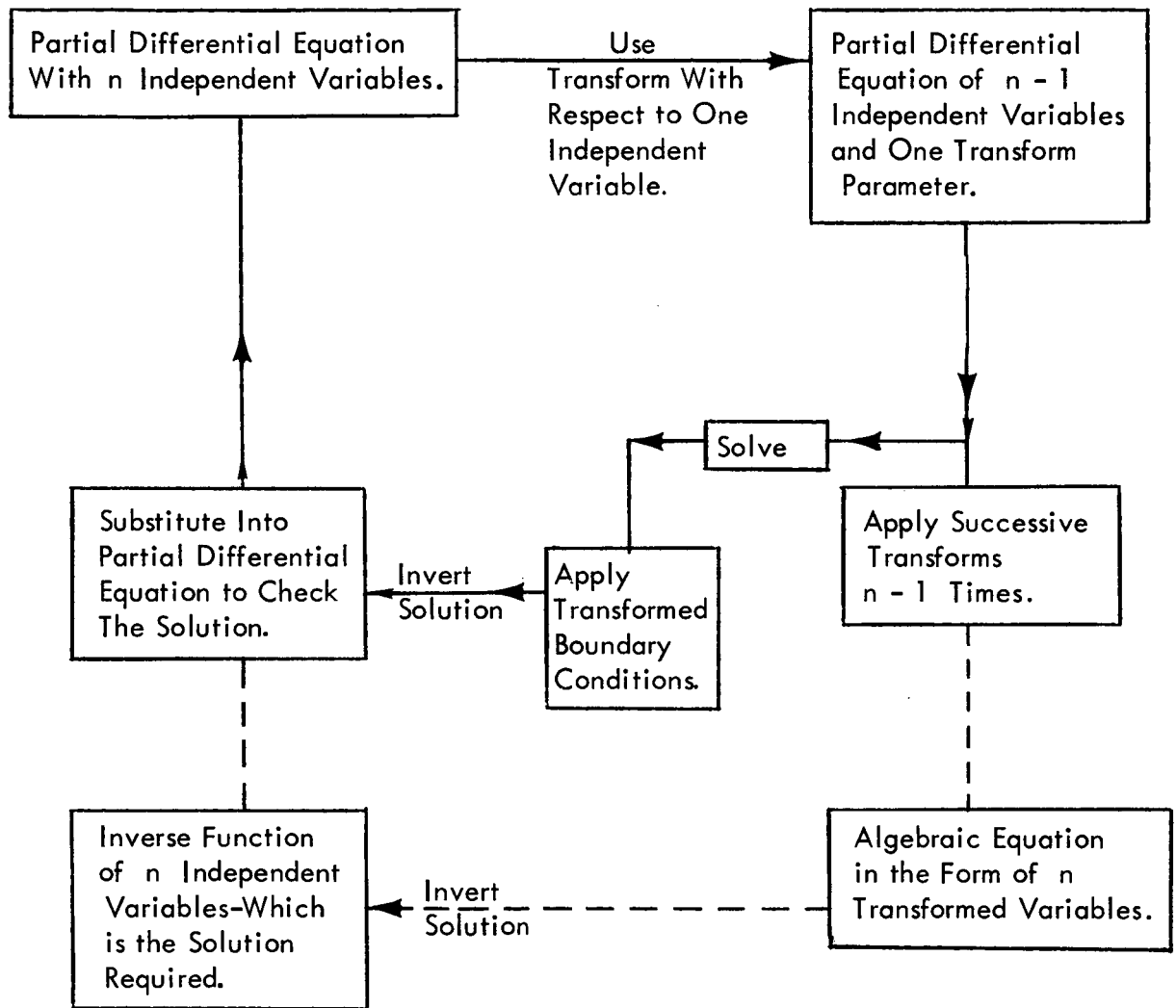
Substitution of (B3), (B4), and (B5) in (4) reduces (4) to

$$\nabla^4 F = E \left[A_{41}^2 \left(\frac{4\pi}{a} \right)^2 \left(\frac{\pi}{b} \right)^2 \right] \left[\cos^2 \frac{4\pi x}{a} \sin^2 \frac{\pi y}{b} - \sin^2 \frac{4\pi x}{a} \cos^2 \frac{\pi y}{b} \right] \quad (B6)$$

APPENDIX C

APPLICATION OF FOURIER FINITE SINE TRANSFORMS WITH RESPECT TO x ON EQUATION (9)

The method of analysis is better explained by the following line diagram .



It is assumed that $F(x, y)$ satisfies the Dirichlet's conditions in the interval $0 \leq x \leq a$.

Then

$$T \left[F(x, y) \right] = \bar{F}(p, y) = \int_0^a F(x, y) \sin \frac{p\pi x}{a} dx \quad (C1)$$

is the definition of Fourier finite sine transform with respect to x .

The inverse of (C1) is

$$F(x, y) = T^{-1} \left[\bar{F}(p, y) \right] = \frac{2}{a} \sum_{p=1}^{\infty} \bar{F}(p, y) \sin \frac{p\pi x}{a} . \quad (C2)$$

Applying (C1) on each term of (9), one obtains

$$\begin{aligned} & T \left[\frac{\partial^4 F}{\partial x^4} \right] + T \left[2 \frac{\partial^4 F}{\partial x^2 \partial y^2} \right] + T \left[\frac{\partial^4 F}{\partial y^4} \right] \\ &= B_1 f_1(y) T \left[\cos^2 \frac{4\pi x}{a} \right] \\ &+ B_1 f_2(y) T \left[\sin^2 \frac{4\pi x}{a} \right] \end{aligned} \quad (C3)$$

Note T , the sine transform with respect to x treats y as a constant.

The transforms of the derivatives of F are obtained as follows, using the method of integration by parts:

$$\begin{aligned} T \left[\frac{\partial^2 F}{\partial x^2} \right] &= \int_0^a \frac{\partial^2 F}{\partial x^2} \sin \frac{p\pi x}{a} dx \\ &= \frac{\partial F}{\partial x} \sin \frac{p\pi x}{a} \Big|_0^a - \beta \int_0^a \frac{\partial F}{\partial x} \cos \frac{p\pi x}{a} dx \end{aligned}$$

where

$$\beta = \frac{p\pi}{a}$$

$$= -\beta^2 \bar{F}(p, y) \quad (C4)$$

$$\text{if } F(a, y) = F(0, y) = 0 .$$

Further note that

$$\begin{aligned} T \left[2 \frac{\partial^4 F}{\partial x^2 \partial y^2} \right] &= 2 \frac{\partial^2}{\partial y^2} \left(T \left[\frac{\partial^2 F}{\partial x^2} \right] \right) \\ &= -2 \beta^2 \frac{\partial^2 \bar{F}(p, y)}{\partial y^2} \end{aligned} \quad (C5)$$

Similarly,

$$\begin{aligned} T \left[\frac{\partial^4 F}{\partial x^4} \right] &= \beta \left[\frac{\partial^2 F}{\partial x^2} (0, y) - (-1)^p \frac{\partial^2 F}{\partial x^2} (a, y) \right] \\ &+ \beta^3 \left[F(0, y) - (-1)^p F(a, y) \right] + \beta^4 \bar{F}(p, y) . \end{aligned} \quad (C6)$$

If

$$\frac{\partial^2 F}{\partial x^2} (a, y) = \frac{\partial^2 F}{\partial x^2} (0, y) = 0$$

and

$$F(a, y) = F(0, y) = 0 ,$$

then

$$T \left[\frac{\partial^4 F}{\partial x^4} \right] = \beta^4 \bar{F}(p, y) . \quad (C7)$$

Using (C4) and (C5) in (C3), (C3) reduces to

$$\beta^4 \bar{F} - 2\beta^2 \frac{\partial^2 \bar{F}}{\partial y^2} + \frac{\partial^4 \bar{F}}{\partial y^4} = B_1 f_1(y) T \left[\cos^2 \frac{4\pi x}{a} \right]$$

$$+ B_1 f_2(y) \tau \left[\sin^2 \frac{4\pi x}{a} \right] \quad (C8)$$

The simplification of the expressions on the right hand side of (C8) is performed in Appendix D.

APPENDIX D

FOURIER FINITE SINE TRANSFORMS WITH RESPECT TO x OF

$$\cos^2 \frac{4\pi x}{a} \text{ AND } \sin^2 \frac{4\pi x}{a} .$$

$$T \left[\cos^2 \frac{4\pi x}{a} \right] = \int_0^a \cos^2 \frac{4\pi x}{a} \sin \frac{p\pi x}{a} dx \quad (D1)$$

$$= \frac{a}{4\pi} \int_0^{4\pi} \cos^2 \bar{x} \sin \frac{p\bar{x}}{4} d\bar{x} \quad (D2)$$

where

$$\bar{x} = \frac{4\pi x}{a} .$$

From p. 142 of Ref. 15 , after considerable simplification, (D2) equals

$$\frac{a}{\pi(8+p)} \left[\left\{ -(-1)^p + 1 \right\} - 2 \left\{ \frac{2}{p} \left[(-1)^p - 1 \right] + \frac{(-1)^p - 1}{\frac{p}{2} - 4} \right\} \right] .$$

Similarly,

$$T \left[\sin^2 \frac{4\pi x}{a} \right] = \int_0^a \sin^2 \frac{4\pi x}{a} \sin \frac{p\pi x}{a} dx \quad (D3)$$

$$= \frac{a}{4\pi} \int_0^{4\pi} \sin^2 \bar{x} \sin \frac{p\bar{x}}{4} d\bar{x} \quad (D4)$$

where

$$\bar{x} = \frac{4\pi x}{a} .$$

From p. 140 of Ref. 15 , after considerable simplification, (D4) equals

$$\frac{-2a}{\pi(8+p)} \left[(-1)^p - 1 \right] \left[\frac{2}{p} + \frac{1}{2(2 - \frac{p}{4})} \right] .$$

APPENDIX E
SOLUTION OF HOMOGENEOUS EQUATION

$$(D^4 - 2\beta^2 D^2 + \beta^4) \bar{F}_1(p, y) = 0$$

Note $D = \frac{\partial}{\partial y}$ and $\beta = \frac{p\pi}{a}$.

The given equation reduces to

$$(D^2 - \beta^2)^2 \bar{F}_1(p, y) = 0 \quad (E1)$$

The roots of the equation are

$$D^2 = \beta^2 \quad \text{repeated twice.} \quad (E2)$$

$$\therefore D = \pm \beta \quad \text{repeated twice.} \quad (E3)$$

$$\therefore \bar{F}_1 = (\bar{C}_1 + \bar{C}_2 y) e^{\beta y} + (\bar{C}_3 + \bar{C}_4 y) e^{-\beta y}$$

$$= C_1 \sinh \beta y + C_2 \cosh \beta y$$

$$+ C_3 y \sinh \beta y + C_4 y \cosh \beta y .$$

APPENDIX F
EVALUATION OF CONSTANTS E_k of (39).

The following constants are assumed for the panel under study.

$$\left. \begin{aligned} E &= 10.5 \times 10^6 \\ a &= 36 \\ b &= 9 \\ \beta &= \frac{\pi}{36} \end{aligned} \right\} \quad \text{Therefore,} \quad (F1)$$

$$\text{If } X = \frac{\beta b}{2}, \quad \text{then} \quad (F2)$$

$$\text{sech } X = 0.9275557714514 \quad (F3)$$

$$\tanh X = 0.3736847478780. \quad (F4)$$

Note

$$B_1 = E \left(\frac{4\pi}{a} \right)^2 \left(\frac{\pi}{b} \right)^2 A_{41}^2. \quad (F5)$$

Further

$$E_1 = \frac{2}{a} \left\{ \frac{-B_1 (\text{sech } X) \left(\frac{2a}{\pi} \right)}{4\beta^4} \right\} \left\{ \frac{\beta b}{2} \tanh X + 2 \right\} \quad (F6)$$

$$E_2 = \frac{B_1 \left(\frac{2a}{\pi} \right)}{4\beta^3} (\text{sech } X) \quad (F7)$$

$$E_3 = \frac{B_1 a}{2 \times 63\pi} \times \frac{62}{\beta^4} \quad (F8)$$

$$E_4 = \frac{B_1 \alpha}{2 \times 63 \pi} (62) \left[- \frac{1}{\left(\beta^2 + \frac{4\pi^2}{b^2} \right)^2} \right] \quad (F9)$$

$$E_5 = \frac{B_1 \alpha}{2 \times 63 \pi} \left(\frac{64}{\beta^4} \right) \quad (F10)$$

$$E_6 = \frac{64 B_1 \alpha}{2 \times 63 \pi} \left(\frac{1}{\left(\beta^2 + \frac{4\pi^2}{b^2} \right)^2} \right) . \quad (F11)$$

APPENDIX G

OPTIMIZATION OF (43) BY RITZ-GALERKIN METHOD.

$$0 = \int_0^a \int_{-\frac{b}{2}}^{+\frac{b}{2}} q \frac{\partial w}{\partial A_{41}} dx dy \quad (G1)$$

$$= \iint \left\{ D \nabla^4 w - h \left[F_{xx} w_{yy} + F_{yy} w_{xx} - 2F_{xy} w_{xy} + \frac{N_{xi}}{h} w_{xx} \right] \right\} \left\{ \sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} \right\} dx dy \quad (G2)$$

$$= \iint \left\{ D \frac{\partial^4 w}{\partial x^4} + 2D \frac{\partial^4 w}{\partial x^2 \partial y^2} + D \frac{\partial^4 w}{\partial y^4} - h F_{xx} w_{yy} - h F_{yy} w_{xx} + 2h F_{xy} w_{xy} - N_{xi} w_{xx} \right\} \left\{ \sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} \right\} dx dy \quad (G3)$$

$$= \iint \left\{ D w_{xxxx} + 2D w_{xxyy} + D w_{yyyy} - h F_{xx} w_{yy} - h F_{yy} w_{xx} + 2h F_{xy} w_{xy} - N_{xi} w_{xx} \right\} \left\{ \sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} \right\} dx dy \quad (G4)$$

$$= \iint \left\{ D A_{41} \left(\frac{4\pi}{a} \right)^4 \sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} + 2D A_{41} \left(\frac{4\pi}{a} \right)^2 \left(\frac{\pi}{b} \right)^2 \sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} + D A_{41} \left(\frac{\pi}{b} \right)^4 \sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} \right\}$$

$$\begin{aligned}
& - h \left(\frac{\pi}{a} \right)^2 \left(\frac{\pi}{b} \right)^2 A_{41} \left(\sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} \right) \left(\sin \frac{\pi x}{a} \right) \left[E_1 \operatorname{ch} \beta y \right. \\
& \quad \left. + E_2 y \operatorname{sh} \beta y + (E_3 + E_5) + (E_4 + E_6) \sin \frac{2\pi}{b} y \right] \\
& + h \left(\frac{4\pi}{a} \right)^2 A_{41} \left(\sin \frac{\pi x}{a} \right) \left[E_1 \beta^2 \operatorname{ch} \beta y + E_2 \beta (y \beta \operatorname{sh} \beta y + 2 \operatorname{ch} \beta y) \right. \\
& \quad \left. - (E_4 + E_6) \left(\frac{2\pi}{b} \right)^2 \sin \frac{2\pi}{b} y \right] \left(\sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} \right) \\
& - 2h \left(\frac{\pi}{a} \right) \left(\frac{4\pi}{a} \right) \left(\frac{\pi}{b} \right) A_{41} \left(\cos \frac{\pi x}{a} \right) \left[E_1 \beta \operatorname{sh} \beta y + E_2 (y \beta \operatorname{ch} \beta y + \operatorname{sh} \beta y) \right. \\
& \quad \left. + (E_4 + E_6) \left(\frac{2\pi}{b} \right) \cos \frac{2\pi}{b} y \right] \left(\cos \frac{4\pi x}{a} \sin \frac{\pi y}{b} \right) \\
& - A_{41} \left(\frac{4\pi}{a} \right)^2 \sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} \left(k \frac{3.84 E h^3}{b^2} \right) \left\{ \sin \frac{4\pi x}{a} \cos \frac{\pi y}{b} dx dy \right. \quad (G5)
\end{aligned}$$

where

ch stands for cosh
and
sh stands for sinh.

Some integrals utilized in (G5) are evaluated below.

$$1) \quad \int_0^a \sin^2 \frac{4\pi x}{a} dx = \frac{a}{2} \quad (G6)$$

$$2) \quad \int_0^a \sin^2 \frac{4\pi x}{a} \sin \frac{\pi x}{a} dx = \frac{64a}{63\pi} \quad (G7)$$

$$3) \quad \frac{1}{2} \int_0^a \cos \frac{\pi x}{a} \sin \frac{8\pi x}{a} dx = \frac{8a}{63\pi} \quad (G8)$$

$$4) \quad \int_{-\frac{b}{2}}^{+\frac{b}{2}} \cos^2 \frac{\pi y}{b} dy = \frac{b}{2} \quad (G9)$$

$$5) \quad \int_{-\frac{b}{2}}^{+\frac{b}{2}} \cos^2 \frac{\pi y}{b} \cosh \beta y dy = \frac{\frac{4\pi^2}{b^2} \sinh \frac{\beta b}{2}}{\beta \left(\beta^2 + \frac{4\pi^2}{b^2} \right)} \quad (G10)$$

$$6) \quad \int_{-\frac{b}{2}}^{+\frac{b}{2}} y \cos^2 \frac{\pi y}{b} \sinh \beta y dy = \left[\frac{1}{2} \frac{b}{\beta} \cosh \frac{\beta b}{2} - \frac{2}{\beta^2} \sinh \frac{\beta b}{2} - \frac{\frac{\beta b}{2} \cosh \frac{\beta b}{2}}{\beta^2 + \frac{4\pi^2}{b^2}} + \frac{2 \left(\beta^2 - \frac{4\pi^2}{b^2} \right)}{\left(\beta^2 + \frac{4\pi^2}{b^2} \right)^2} \sinh \frac{\beta b}{2} \right] \quad (G11)$$

$$\begin{aligned}
 6a) \quad & \int_{-\frac{b}{2}}^{+\frac{b}{2}} y \cos^2 \frac{\pi y}{b} \sinh \beta y \, dy \\
 &= \frac{2\pi^2 \cosh \frac{\beta b}{2}}{b\beta \left(\beta^2 + \frac{4\pi^2}{b^2}\right)} - \frac{2 \sinh \frac{\beta b}{2}}{\beta^2 \left(\beta^2 + \frac{4\pi^2}{b^2}\right)^2} \left[\frac{12\pi^2 \beta^2}{b^2} + \frac{16\pi^4}{b^4} \right] \quad (G11a)
 \end{aligned}$$

$$7) \quad \int_{-\frac{b}{2}}^{+\frac{b}{2}} \cos^2 \frac{\pi y}{b} \sin \frac{2\pi y}{b} \, dy = 0 \quad (G12)$$

$$8) \quad \int_{-\frac{b}{2}}^{+\frac{b}{2}} \sin \frac{2\pi y}{b} \sinh \beta y \, dy = \frac{4\pi}{b \left[\beta^2 + \frac{4\pi^2}{b^2} \right]} \sinh \frac{\beta b}{2} \quad (G13)$$

$$\begin{aligned}
 9) \quad & \frac{1}{2} \int_{-\frac{b}{2}}^{+\frac{b}{2}} y \sin \frac{2\pi y}{b} \cosh \beta y \, dy \\
 &= \left[\frac{\pi}{\beta^2 + \frac{4\pi^2}{b^2}} \right] \left[-\cosh \frac{\beta b}{2} + \frac{\frac{4\beta}{b}}{\beta^2 + \frac{4\pi^2}{b^2}} \sinh \frac{\beta b}{2} \right] \quad (G14)
 \end{aligned}$$

$$10) \quad \int_{-\frac{b}{2}}^{+\frac{b}{2}} \cos^2 \frac{\pi y}{b} \sin \frac{2\pi y}{b} \, dy = 0$$

Let us concentrate our attention on two different thicknesses of the plate, namely, $h = 0.040''$ (case 1) and $h = 0.032''$ (case 2). The page is divided into two sections, when the values are different.

Case 1	Case 2
$h = 0.040''$	$h = 0.032''$
$D = 63 \text{ lb. in.}$	$D = 32.256 \text{ lb. in.}$

The integrals of (G5) are evaluated exactly in a closed form, utilizing (G6) to (G15). The coefficients of A_{41} and A_{41}^3 are listed below.

Case 1	Case 2
303.0505 A_{41}	155.1619 A_{41}
- 107.1129 $\times 10^6 A_{41}^3$	- 856.9028 $\times 10^5 A_{41}^3$
437.1314 $\times 10^6 A_{41}^3$	349.7051 $\times 10^6 A_{41}^3$
- 544.2502 $\times 10^4 A_{41}^3$	- 435.4001 $\times 10^4 A_{41}^3$
- 314.4237 $k A_{41}$	- 160.9849 $k A_{41}$

Note k stands for the ratio, total load on the panel over Euler critical load of the panel.

TABLE I

NON-DIMENSIONAL BUCKLE DEPTH

For various in-plane compressive loads, expressed as multiples of buckling load, investigated for two different values of plate thickness of 0.040" and 0.032".

k	$\frac{w}{h}$
1	0.0044
2	0.025
3	0.035
5	0.049
10	0.074
25	0.12
50	0.17
100	0.25
200	0.35

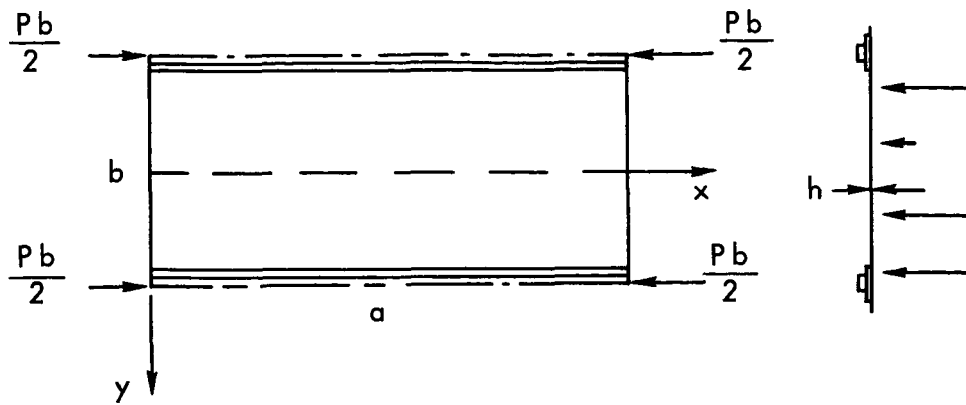


Figure 1. The specimen under study and its coordinate system.

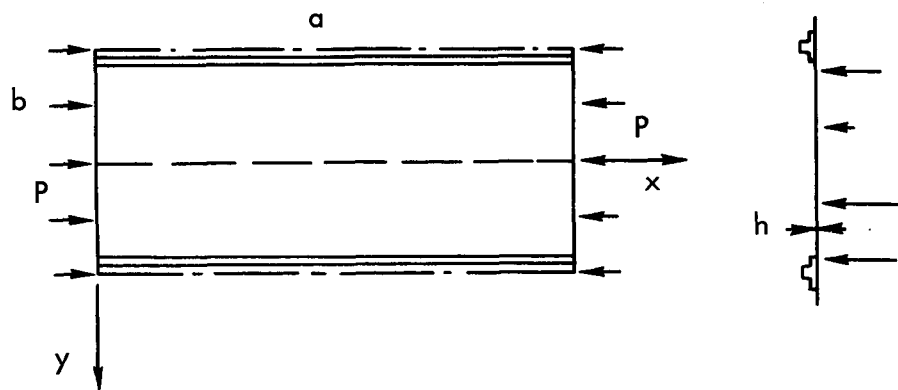


Figure 2. The problem to be analyzed in this study.

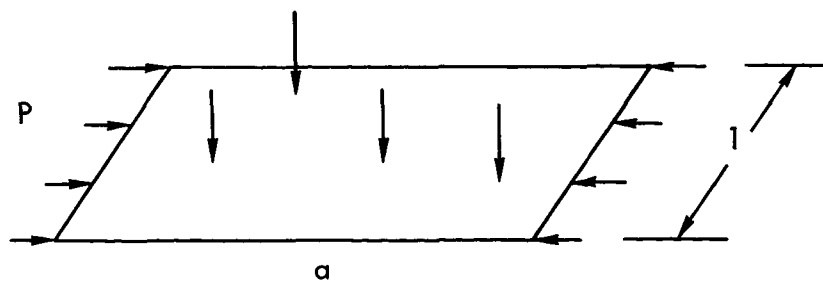


Figure 3. A plate element of unit width under lateral and in-plane loadings.

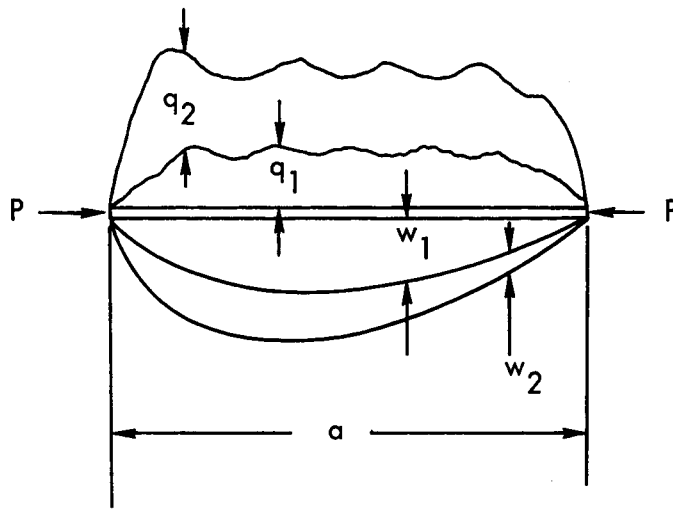


Figure 4. The end view of a plate element under an axial end load P and lateral acoustic load $q_1 + q_2$.

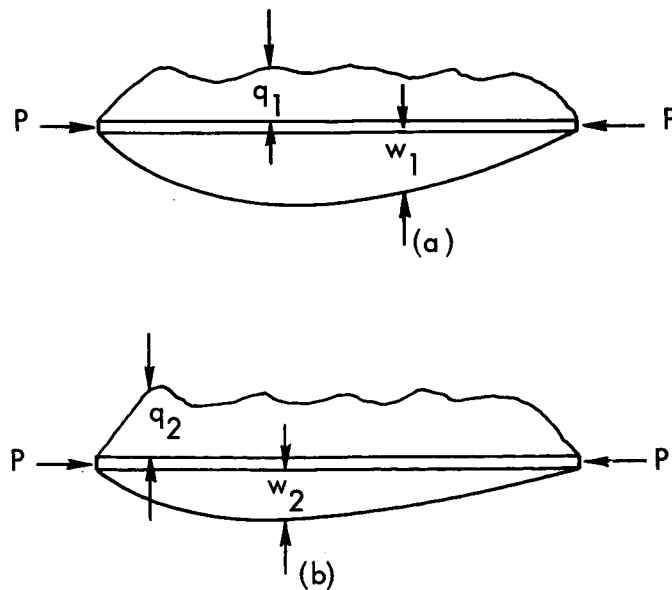


Figure 5. The system and the associated response of Figure 5 are equivalent to those of Figure 4.

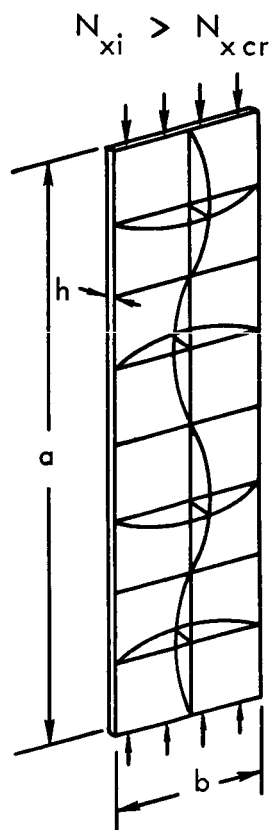


Figure 6. The buckled plate

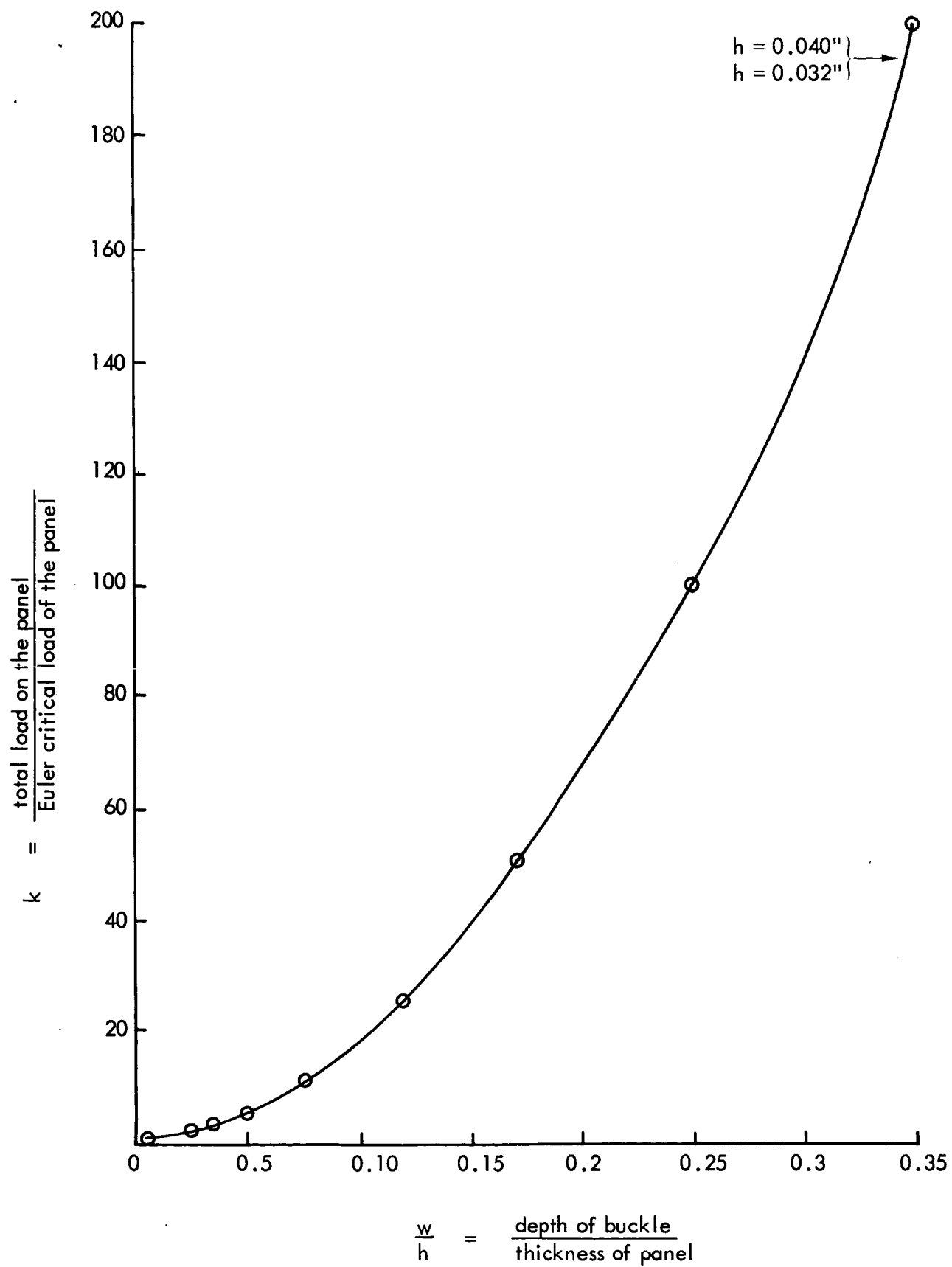


Figure 7. Non-dimensional compressive load versus non-dimensional buckle depth.